Coupled State Policy Dynamics in Evolutionary Games

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Chapter 6

Coupled state policy dynamics in evolutionary games

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In this work we want to extend the EGT models by introducing the concept of individual state. We analyze a particular simple case, in which we associate a state to each player, and we suppose that this state determines the set of available actions. We consider deterministic stationary policies and we suppose that the choice of a policy determines the fitness of the player and it impacts the evolution of the state. We define the interdependent dynamics of states and policies and we introduce the State Policy coupled Dynamics (SPcD) in order to study the evolution of the population profile and we prove the relation between the rest points of our system and the equilibria of the game. We then assume that the processes of states and policies move with different velocities: this assumption allows us to solve the system and then to find the equilibria of our game with two different methods: the singular perturbation method and a matrix approach.

This chapter is organized as follows: in Section 6.1 we briefly present standard EGT main definitions and results. In Section 6.2 we extend EGT problems introducing individual states. We define the notion of equilibrium profile of the population in such context and the dynamics of states and policies. We show the relation between the rest points of the system and the equilibria of the game. We then provide, in section 6.3, the two different methods to solve this system, which can be applied when assuming a two-timescales behavior. We conclude with some numerical results.

6.1 Standard Evolutionary Game Theory

6.1.1 Evolutionary Stable Strategy

Consider an infinitely large population of players, where each player repeatedly meets a randomly selected individual within the population. Each individual disposes of a finite pure action space $\mathcal{A}$, $|\mathcal{A}| = K$. Let $\Delta(\mathcal{A}) = \{p \in \mathbb{R}_+^K \sum_{i\in\mathcal{A}} p_i = 1\}$ be the set of mixed strategies, that are probability measures over the action space. We define by $F(p, q)$ the expected payoff of an individual playing p against an opponent using q, where $p, q \in \Delta(\mathcal{A})$. If A is the payoff matrix associated to the pairwise interactions, then $F(p, q) = p^T Aq$.

An ESS is a strategy that, if adopted by the whole population, it is resistant against mutations of a small fraction of individuals in the population. Suppose that the whole population adopts a strategy q, and that a fraction ϵ of *mutants* deviate to strategy p. Strategy q is an ESS if $\forall p \neq q$, there exists some $\epsilon_p > 0$ such that:

$$\forall \epsilon \in (0, \epsilon_p) \qquad F(q, \epsilon p + (1-\epsilon)q) > F(p, \epsilon p + (1-\epsilon)q) \tag{6.1}$$

The following proposition allows to characterize an ESS through its stability properties.

Proposition 20. *$q \in \Delta(\mathcal{A})$ is an ESS if and only if it satisfies the following conditions:*

- *Nash Condition: $F(q, q) \geq F(p, q) \qquad \forall p$,*
- *Stability Condition: $F(q, q) = F(p, q) \Rightarrow F(q, p) \geq F(p, p) \qquad \forall p \neq q$.*

It immediately follows that any strict Nash equilibrium is an ESS,while the converse is not true. When players are programmed to pure actions and mixed ones are not allowed, the notion of ESS is associated to the state of the population instead that to a mixed action, as introduced by Taylor and Jonker in [255]. The population state is given by the vector $q = (q_1, \ldots, q_K)$, where $q_i = \frac{N_i}{N}$ is the proportion of individuals in the population playing pure action i. We observe that $q \in \Delta(\mathcal{A})$, so it is formally equivalent to a mixed action in $\Delta(\mathcal{A})$. Let $F(q,p)$ denotes the immediate expected payoff of a group of individuals in state q playing against a population in state p, and let $F(i,p)$ denotes the immediate expected payoff of an individual playing pure strategy i against a population in state p. We have that: $F(q,p) = \sum_{i=1}^{K} q_i F(i,p)$.

The definition of the ESS concerning states is then equivalent to that of the ESS concerning strategies: state q is an ESS if $\forall p \neq q$, there exists some $\epsilon_p > 0$ such that $F(q, \epsilon p + (1-\epsilon)q) > F(p, \epsilon p + (1-\epsilon)q)$, $\forall \epsilon < \epsilon_p$.

6.1.2 Replicator Dynamics

Let the population be of large but finite size N and let players be programmed to pure actions $\mathcal{A}$. Let N_i be the number of individuals adopting $i \in \mathcal{A}$. The population profile at time t is given by the vector $q(t) = (q_1(t), q_2(t), \ldots, q_K(t))$, $q(t) \in \Delta(\mathcal{A})$, where $q_i = \frac{N_i}{N}$ is the fraction of individuals playing pure action i. The replicator dynamics describes how the distribution of pure actions evolves in time depending on interactions between individuals. Replicator dynamics of action $i \in \mathcal{A}$ is expressed by the following equation:

$$\dot{q}_i(t) = q_i(t)(F_i(q(t)) - \bar{F}(q(t))), \tag{6.2}$$

where $F_i(q(t))$ is the immediate fitness of an individual playing i and $\bar{F}(q(t))$ is the average immediate fitness of the population. In the two-actions case, with $\mathcal{A} = \{x, y\}$, if $q(t)$ indicates the share of the population playing action x at time t, the latter equation can be rewritten as follows: $\dot{q}(t) = q(t)(1-q(t))(\bar{F}_x(q(t)) - \bar{F}_y(t))$.

The replicator equation has numerous properties and there is a close relationship between its rest points and the equilibria. The *folk theorem of evolutionary game theory* [147] states that:

1. any Nash equilibrium is a rest point of the replicator equation;
2. if a Nash equilibrium is strict then it's asymptotically stable;
3. if a rest point is the limit of an interior orbit for $t \to \infty$, then it is a Nash equilibrium;

4. any stable rest point of the replicator dynamics is a Nash equilibrium.

Any ESS is an asymptotically stable rest point and an interior ESS is globally stable, but the converse does not hold in general.

6.2 Individual State in EGT framework

6.2.1 Individual State and its dynamics

In this section, we introduce the concept of *individual state* in EGT framework and we present the consequent dynamics of our model. We consider a population of N individuals, where each individual can be in one of two possible states, $\mathcal{S} = \{1, 0\}$; every individual goes through a cycle that starts at state 1, moves to states 0 after some random time at a rate that depends on its policy. After some exponentially distributed time it returns to state 1 and so on. At each pairwise interaction, the set of available actions of a player depends on its state: in state 1, $\mathcal{A}_1 = \{x, y\}$, whereas in state 0 an individual can only use y.

We consider the set of deterministic stationary policies, which are functions that associate to each state an action in $\mathcal{A}$ and do not depend on time. Let u_x (resp. u_y) be the deterministic stationary policy which consists in always playing action x (resp. y) in state 1. In state 0, an individual always plays y. Each player chooses one deterministic stationary policy and we denote by $q_x(t)$ the proportion of individuals in the population that play the deterministic stationary policy u_x at time t. In the same way, $q_y(t)$ denotes the proportion of individuals that play the deterministic stationary policy u_y at time t and we have $q_y(t) = 1 - q_x(t)$.

We suppose that the policy chosen impacts the utility of the player and also the time he spends in state 1. We define by μ_i the rate of decay from state 1 to state 0 when using policy u_i, $i \in \{x, y\}$, where $\mu_x > \mu_y$, and by μ the rate of change from state 0 to state 1. The rates depend on the action induced by the policy chosen by the player. Then, by abuse of description, we say that the rates are controlled by the policies, but in fact, they are controlled by the actions induced by the policies.

We define the proportion of individuals that are in state 1 at time t as $p_1 \equiv \frac{N_1}{N} = \frac{1}{N}\sum_{i=1}^{N} \nVdash_{\{x_i=1\}}$, where x_i denotes the state of player $i \in \{1, \ldots, N\}$ and $\nVdash_{\{x_i=1\}} = 1$ when $x_i = 1$ and it's zero otherwise. We denote by Y_i the random variable $Y_i \equiv \nVdash\{x_i = 1\} \in \{0, 1\}$. Y_i has a Bernoulli distribution, such that $\mathbb{P}(Y_i = 1) = 1 - \mathbb{P}(Y_i = 0) = p_1^i$, and thus $\mathbb{E} = p_1^i$. The individual state dynamics can be described by the following differential equation:

$$\dot{p}_1^i(t) = -\mu_x p_1^i(t) q_x(t) + \mu(1 - p_1^i(t)) - \mu_y p_1^i(t)(1 - q_x(t)),$$

If each individual plays policy u_x with a probability q_x, $(Y_i)_i$ are i.i.d. and thus, for the strong law of large numbers, when $N \to \infty$, $p_1 \to p_1^i$ and consequently:

$$\dot{p}_1(t) = -\mu_x p_1(t) q_x(t) + \mu(1 - p_1(t)) - \mu_y p_1(t)(1 - q_x(t)), \tag{6.3}$$

where $q_x(t)$ is the proportion of individuals in the population that play the deterministic stationary policy u_x at time t.

6.2.2 Individual fitness

At each pairwise interaction, the immediate fitness obtained by an individual depends on his current action and the current action of its opponent. It is given by the following fitness matrix:

$$A := \begin{array}{c} \\ x \\ y \end{array} \begin{array}{c} \begin{array}{cc} \quad x & \quad\quad y \end{array} \\ \begin{pmatrix} (a,a) & (b,c) \\ (c,b) & (d,d) \end{pmatrix} \end{array} \tag{6.4}$$

where x and y are the available actions and the matrix entry $A_{i,j}$ indicates the payoff respectively of the first (row) and the second (column) player. The immediate expected fitness of a player interacting at time t, depends on the population profile at that time. As we added a state component in our framework, the population profile at time t is now expressed by the couple $\xi(t) := (p_1(t), q_x(t))$.

We define $\bar{J}_i^1(\xi(t))$ (resp. $\bar{J}_i^0(\xi(t))$) as the immediate expected fitness of an individual who is in state 1 (resp. 0) and plays deterministic stationary policy u_i against a population whose profile is $\xi(t)$. It follows that the value of $\bar{J}_x^1(\xi(t))$ corresponds to the immediate expected fitness of an individual playing action x, against a population whose profile is $\xi(t)$. We observe that $\bar{J}_y^1(\xi(t)) = \bar{J}_y^0(\xi(t)) = \bar{J}_x^0(\xi(t))$, being all equals to the value of the immediate expected fitness of an individual playing action y; we thus denote it simply as $\bar{J}_y(\xi(t))$. We obtain the following expressions:

$$\bar{J}_x^1(\xi(t)) := p_1(t)(q_x(t)a + (1 - q_x(t))b) + (1 - p_1(t))b,$$

$$\bar{J}_y(\xi(t)) := p_1(t)(q_x(t)c + (1 - q_x(t))d) + (1 - p_1(t))d.$$

The immediate expected fitnesses $\bar{F}_i(\xi(t))$ at time t of an individual playing deterministic stationary policy u_i, $i \in \{x, y\}$, are given by:

$$\bar{F}_x(\xi(t)) = p_1^i(t)\bar{J}_x^1(\xi(t)) + (1 - p_1^i(t)))\bar{J}_y(\xi(t))$$
$$\bar{F}_y(\xi(t)) = \bar{J}_y(\xi(t)).$$

The average expected fitness of the whole population with profile $\xi(t) = (p_1(t), q_x(t))$ is defined as:

$$\bar{F}(\xi(t)) = q_x(t)\bar{F}_x(\xi(t)) + (1 - q_x(t))\bar{F}_y(\xi(t)).$$

6.2.3 Evolutionary Stable Strategies

We want to study the properties of stability of the population profile, supposing that individuals can play only deterministic policies.

We then define the equilibrium profile of the population as follows:

Definition 11 (Equilibrium profile)**.** *A population profile $\xi^* = (p_1^*, q_x^*)$ is an equilibrium profile if $\forall u_i \in supp(q_x^*)$ we have that:*

$$\bar{F}_i(\xi^*) \geq \bar{F}_j(\xi^*) \qquad j \neq i,$$

where $supp(q_x^*) = \{u_i,\ i \in \mathcal{A} | q_i^* > 0$ given the state $q_x^*\}$.

Proposition 21. *If the population profile $\xi^* = (\tilde{p}_1^*, \tilde{q}_x^*)$ satisfies the indifference principle:*

$$\bar{F}_x(\xi^*) = \bar{F}_y(\xi^*),$$

then it is an equilibrium profile.

We now look for the stability condition of the equilibrium state. This leads to the formal definition of the ESS: in our model it corresponds to an evolutionary stable population profile. Given $\tilde{q}_x \in [0, 1]$ and a population profile $\xi = (p_1, q_x)$, the average fitness of a group of individuals such that a proportion $\tilde{q}_x$ of the group play stationary deterministic policy u_x against a population whose profile is $\xi = (p_1, q_x)$ is given by:

$$\bar{F}_{\tilde{q}_x}(\xi) = \tilde{q}_x\bar{F}_x(\xi) + (1 - \tilde{q}_x)\bar{F}_y(\xi).$$

Definition 12 (ESS)**.** *A population profile $\xi^* = (p_1^*, q_x^*)$ is an ESS if $\forall \xi = (p_1, q_x)$, the two following conditions hold:*

1. $\bar{F}_{q_x^*}(\xi^*) \geq \bar{F}_{q_x}(\xi^*)$,
2. $\bar{F}_{q_x^*}(\xi^*) = \bar{F}_{q_x}(\xi^*) \Rightarrow \bar{F}_{q_x^*}(\xi) \geq \bar{F}_{q_x}(\xi)$.

6.2.4 Policy Based Replicator Dynamics

As we focus here on policies instead that on strategies, we introduce a policy based replicator dynamics (PbRD), to study the evolution of the share of individuals $q_x(t)$ using pure policy u_x at time t. The PbRD is given by the following equation:

$$\dot{q}_x(t) := q_x(t)(\bar{F}_x(\xi(t)) - \bar{F}(\xi(t))). \tag{6.5}$$

Then, the growth rate of the population share using policy u_x is:

$$\frac{\dot{q}_x(t)}{q_x(t)} = \bar{F}_x(\xi(t)) - \bar{F}(\xi(t)), \tag{6.6}$$

The PbRD can be written as:

$$\begin{array}{rcl} \dot{q}_x(t) & = & q_x(t)(1 - q_x(t))(\bar{F}_x(\xi(t)) - \bar{F}_y(\xi(t))), \\ & := & g(p_1(t), q_x(t)). \end{array}$$

We now investigate the dynamics of actions, where the fitness is a function of the population profile depending on policies and states. If we pick one random individual in the population at time t, the probability that he plays pure action x, is given by the product $q(t) = q_x(t)p_1(t)$, which leads to:$\dot{q}(t) = \dot{q}_x(t)p_1(t) + q_x(t)\dot{p}_1(t)$. By carrying out the expression of $\dot{q}_x(t)$, after some basic algebra, we get the following equation for the growth rate of the proportion of individuals playing action x in the population at time t:

$$\frac{\dot{q}(t)}{q(t)} = (\bar{F}_x(\xi(t)) - \bar{F}(\xi(t))) + \frac{\dot{p}_1(t)}{p_1(t)} \tag{6.7}$$

Equation (6.7) shows how the evolution of states impacts the dynamics of actions in our context. The growth rate of action x is increasing in the growth rate of state 1. We observe that a sufficiently high growth rate of state 1 can leads to a growing rate of action x even if policy u_x is non-optimal.

6.2.5 State-Policy Coupled Dynamics

The replicator dynamics of our model are defined by the following system of State-Policy Coupled Dynamics (SPcD) which combines the dynamics of the individual state and the dynamics of the policies used in the population:

$$(S) \left\{ \begin{array}{l} \dot{p}_1 = h(\xi(t)) \\ \dot{q}_x = g(\xi(t)) \end{array} \right.$$

where $\xi(t) = (p_1(t), q_x(t))$ corresponds to the population profile. The rest points of the SPcD is the couple $\xi^* = (p_1^*, q_x^*)$ satisfying:

$$\begin{cases} h(\xi^*) = 0 \\ g(\xi^*) = 0. \end{cases} \tag{6.8}$$

Lemma 1. *Any interior rest point of the SPcD* (S) *is a state equilibrium of the state-policy game.*

Remark 2. *Note that the converse does not necessarily hold. Any equilibrium state is a rest point of the PbRD in (6.5), but it's not necessarily a rest point of the individual state dynamics.*

Lemma 2. *Any stable rest point of the SPcD* (S) *is an equilibrium profile of the state-policy game.*

Finally, in order to guarantee that a rest point is an ESS, we need more properties on the rest point of the SPcD. A sufficient conditions to guaratee evolutionarly stability of a rest point is the strong stability [147]. Another method to verify that a rest point ξ^* is an ESS is to construct a suitable local Lyapunov function [142] for the raplicator dynamics in ξ^*.

6.3 Two time-scales behavior

We assume here that the state and the policy dynamics move with different velocities. The individual state dynamics, given by equation (6.3), are supposed to move very fast compared to the slow updating strategy process modeled through the SPcD (6.3). This assumption allows us to find the equilibrium profile of the population with two different approaches. As we showed the relation between the equilibria of the game and the rest points of the SPcD, we can solve the system (S) to find the equilibria. Alternatively, we can consider the stationary distribution of states and rewrite our model as a matrix game.

6.3.1 Singular perturbations

If we consider the two-time-scales behavior of the system (S), we can approximate its solution using the standard Singular Perturbation Model [164] to find the rest points of the SPcD. We introduce the parameter $\epsilon > 0$, such that :

$$\epsilon \dot{p}_1 := h(p_1, q_x).$$

We rewrite the system of the two coupled differential equations:

$$(S_\epsilon)\begin{cases} \epsilon\dot{p}_1 = h(p_1, q_x), \\ \dot{q}_x = g(p_1, q_x). \end{cases}$$

The parameter ϵ is a small positive scalar which serves to represent the different timescales of the two processes, where the velocity of the state process, $\dot{p}_1 = h(p_1, q_x)/\epsilon$, is fast when ϵ is small.

The theory of singular perturbed differential equations gives an easy way to solve an approximation of the system when $\epsilon \to 0$. We can consider the *quasi-steady-state-model* [164] by first solving in p_1 the transcendental equation $0 = h(p_1, q_x)$ and then rewriting the differential equation $\dot{q}$ as a function of the obtained roots. As the latter equation has a unique real solution $\bar{p}_1 := \pi_1(q_x)$, our system is in *normal form* . This allows us to solve the second differential equation called the quasi-steady-state equation:

$$\dot{q}_x = g(\pi_1(q_x), q_x). \tag{6.9}$$

As the assumption [164] $\frac{\partial h}{\partial p_1}(p_1, q_x) < 0$ is satisfied, the reduced model is a good approximation of the original system. The two-time-scale behavior of $p_1(t)$ and $q_x(t)$ has a geometric interpretation, as trajectories in $\mathbb{R}^2$. If we define the manifold sets $M_\epsilon := \{\phi \quad \text{s.t.} \quad p_1 = \phi(q_x, \epsilon) \quad \text{and} \quad \epsilon = h(q_x, \phi(q_x, \epsilon))\}$, it is possible to rewrite the problem in terms of invariant manifolds. When $\epsilon = 0$, the manifold M_0 corresponds to the expression of the quasi steady state model. When the condition $\frac{\partial h}{\partial p_1}(p_1, q_x) < 0$ is satisfied, we have that the equilibrium manifold M_0 is stable (attractive). Particularly, the important result is that the existence of a conditionally stable manifold M_0 for $\epsilon = 0$ implies the existence of an invariant manifold M_ϵ satisfying the following convergence for all $\epsilon \in [0, \epsilon^*]$:

$$\phi(\epsilon, q_x) \to \phi(0, q_x), \quad \text{and} \quad M_\epsilon \to M_0 \quad \text{as} \quad \epsilon \to 0.$$

The positive constant ϵ^* is determined such that the following manifold condition is satisfied:

$$\epsilon \frac{\partial \phi}{\partial x} g(\phi(q_x, \epsilon), q_x) = h(\phi(q_x, \epsilon), q_x),$$

for all q_x and $\epsilon \in [0, \epsilon^*]$. The attractiveness of the slow manifold M_0 is illustrated in the numerical illustrations section. Let us now compute the solution of the approximate system (S_0). We then consider the stationary regime of the individual state dynamics (expressed by Equation (6.3)). By imposing $\dot{p}_1 = 0$, we obtain the following slow manifold $M_0 := \{\phi \quad \text{s.t.} \quad p_1 = \phi(q_x, 0) \quad \text{and} \quad 0 = h(q_x, \phi(q_x, 0))\}$:

$$\phi(q_x, 0) = \frac{\mu}{\mu + \mu_x q_x + \mu_y(1 - q_x)} := \pi_1(q_x). \tag{6.10}$$

The PbRE (6.5) can now be rewritten as:

$$\begin{aligned} \dot{q}_x(t) &= q_x(t)(1 - q_x(t)) \times \dots \\ &\quad \left[\bar{F}_x(\pi_1(q_x(t)), q_x(t)) - \bar{F}_y(\pi_1(q_x(t)), q_x(t))\right]. \end{aligned}$$

Proposition 22. *Considering the singular perturbations method with $\epsilon \to 0$, the solutions of the coupled differential equations (S_0) is the population profile $\xi^* = (p_1^*, q_x^*)$, such that:*

$$p_1^* = \frac{\mu - s^*(\mu_x - \mu_y)}{\mu + \mu_y} \quad \text{and} \quad q_x^* = \frac{s^*(\mu + \mu_y)}{\mu - s^*(\mu_x - \mu_y)}, \tag{6.11}$$

where s^* is the equilibrium of the standard replicator dynamics (6.2) when considering payoff matrix (6.4):

$$s^* = \frac{d - b}{\Delta} \quad \text{with} \quad \Delta = a - b - c + d.$$

Note that the rest point q_x^* of the PbRE (6.5) verifies:

$$q_x^* \pi_1(q_x^*) = s^*.$$

This result says that the equilibrium probability that any individual picked out randomly in the population, is playing action x is equal to s^*. This value is the mixed equilibrium of the standard matrix game. It means that, if we consider a state dependent action game, the equilibrium is obtained under conditional probability over the state.

We have the following necessary and sufficient condition under which the solution obtained is a strict interior point.

Lemma 3. *The solution q_x^* obtained in proposition (22) is a strict interior point if and only if:*

$$\mu > \mu_x \frac{s^*}{1 - s^*}.$$

An important remark is that this condition does not depend on the rate μ_y.

In the next section, we present an alternative method based on rewriting our game problem into a matrix game considering only pure policies.

6.3.2 Matrix Approach

The two time-scales assumption implies that we can consider individuals in stationary state. We can thus rewrite our model as a matrix game in which individuals play stationary deterministic policies instead of actions. We get the following bimatrix game:

$$\begin{array}{cc} & \begin{array}{cc} \qquad u_y \qquad & \qquad u_x \qquad \end{array} \\ \begin{array}{c} u_y \\ u_x \end{array} & \begin{pmatrix} (J(u_y,u_y),J(u_y,u_y)) & (J(u_y,u_x),J(u_x,u_y)) \\ (J(u_x,u_y),J(u_y,u_x)) & (J(u_x,u_x),J(u_x,u_x)) \end{pmatrix} \end{array}$$

where $J(u_i, u_j)$ is the expected average fitness of an individual playing pure policy u_i against an individual using u_j, $i, j \in \{x, y\}$. The stationary distributions in states 1 and 0 are given by the following time ratios:

$$\bar{T}_1(i) = \frac{\frac{1}{\mu_i}}{\frac{1}{\mu} + \frac{1}{\mu_i}} = \frac{\mu}{\mu + \mu_i} \quad \text{and} \quad \bar{T}_0(i) = \frac{\mu_i}{\mu + \mu_i},$$

where $i \in \mathcal{A}$ denotes the choice of policy u_i. The expected average fitness $J(u_i, u_j)$ can thus be expressed as a function of the time ratios:

$$J(u_i, u_j) \quad = \quad \sum_{s,s' \in \mathcal{S}} \sum_{a,a' \in \mathcal{A}} T_s(a) T_{s'}(a') R(a, a'),$$

where $R(a, a')$ is the immediate fitness of a player using action a against an opponent playing a'. We consider the payoffs bimatix (6.4) and we thus obtain:

$$\begin{aligned} & J(u_y,u_y) = d, \quad J(u_x,u_y) = T_1(x)b + T_0(x)d, \\ & J(u_y,u_x) = T_1(x)c + T_0(x)d, \\ & J(u_x,u_x) = T_1(x)\left[T_1(x)a + T_0(x)b\right] \\ & \qquad\qquad\quad + T_0(x)\left[T_1(x)c + T_0(x)d\right]. \end{aligned}$$

If we consider this matrix game as a representation of a standard evolutionary game, we can write the replicator dynamics equation for this evolutionary game as:

$$\dot{\delta}(t) = \delta(t)(1 - \delta(t))(J(u_x, \delta(t)) - J(u_y, \delta(t))), \tag{6.12}$$

where $\delta(t)$ is the proportion of individuals in the population who play pure policy u_x at time t. The standard replicator dynamics equation for this matrix game, if it converges, converges to the ESS of the evolutionary game. The mixed equilibrium δ^* for this matrix game is obtained by solving the indifference principle equation:$J(u_y, \delta^*) = J(u_x, \delta^*)$, where $J(u_i, q) = (1-q)J(u_i, u_y) + qJ(u_i, u_x)$ with $i \in \mathcal{A}$. We know from standard evolutionary game theory that the mixed equilibrium is expressed by:

$$\delta^* = \frac{J(u_y, u_y) - J(u_x, u_y)}{J(u_x, u_x) - J(u_y, u_x) + J(u_y, u_y) - J(u_x, u_y)}.$$

Thus, after some algebras, the ESS obtained by rewriting our game by considering stationary deterministic policies and average fitnesses is:

$$\delta^* = \frac{s^*}{\bar{T}_F(x)}. \tag{6.13}$$

6.3.3 Relations between equilibria

We Now compare the two equilibria we obtained by considering two different point of view of the problem. In section 6.3.1, we supposed that each individual plays a deterministic policy u_x which consists in always choosing action x in state F and, by applying the singular perturbation method, we have been able to determine the equilibrium of such a game. In section 6.3.2, we assumed that individuals are in their stationary states and we rewrite the game as a standard evolutionary game.

Proposition 23. *The relation between the equilibrium δ^* and the equilibrium q_x^* is the following:*

$$q_x^* < \delta^*.$$

We are able also to compare the two equilibria in terms of average fitness obtained by the population, i.e. $J(\delta^*, \delta^*)$ and $\bar{F}(q_x^*, p_1^*)$.

Proposition 24. *The average fitnesses of the population at the two equilibria points obtained with the two approaches are equals, i.e. $\bar{F}(q_x^*, p_1^*) = J(\delta^*, \delta^*)$.*

Finally, we prove that the two mixed strategies obtained with the two approaches are not equal, but are in the same equivalent class in terms of occupation measures. We denote by $\bar{T}_1(q)$ the average sojourn time in state F for an individual who plays mixed strategy q. This mixed strategy has two possibilities. First, it can be a mixed policy between the pure policies u_y and u_x, like proposed in the section 6.3.2. Second, a mixed policy characterized

by a probability q to play action x in state F, at each time an individual is in state F. This second point of view is proposed in section 6.3.1. The two equilibria obtained by the singular perturbations method and the matrix game reformulation are in the same equivalent class in terms of the occupation measures. It means that they should satisfy: $\bar{T}_1(\delta^*) = \bar{T}_1(q_x^*)$.
We have for the case of pure policies:

$$\bar{T}_1(\delta^*) = \delta^* \frac{\mu}{\mu + \mu_x} + (1 - \delta^*) \frac{\mu}{\mu + \mu_y}. \tag{6.14}$$

For the case of mixed policies, we have:

$$\bar{T}_1(q_x^*) = \pi_1(q_x^*) = \frac{\mu}{\mu + \mu_x q_x^* + \mu_y (1 - q_x^*)}. \tag{6.15}$$

This important result is proved in the following proposition.

Proposition 25. *The mixed equilibrium δ^* over the pure policies and the equilibrium obtained by the singular perturbation approach yield to the same occupation measures, i.e.*

$$\bar{T}_1(\delta^*) = \bar{T}_1(q_x^*).$$

This two previous results show that we can define two equivalent classes for deterministic stationary policies that yield same average fitness and occupation measures. This also leads us to generalize to several states and actions.

6.4 Numerical Illustrations

We illustrate here the theoretical results obtained in previous sections with numerical solutions. We consider a first numerical example with the following transition rates: $\mu = 10$, $\mu_x = 1.5$ and $\mu_y = 1$. The fitnesses of the matrix game are: $a = -0.3$, $c = 0$, $b = 1$ and $d = 0.5$. Those values yield to the following equilibrium of the standard evolutionary game $s^* = \frac{5}{8} = 0.625$.

We plot on figure **??** the trajectories of the system (S_ϵ) of the coupled differential equations for different initial conditions and for $\epsilon = 0.01$. We simulate a discrete time version of the differential equations. We plot also the invariant manifold M_0 and we observe that it is an attractor of the trajectories.

More, based on proposition (22), we have the following solution of the system, by considering the singular perturbation method based on the steady-state model:

$$q_x^* = 0.7097, \quad \text{and} \quad p_1^* = 0.8807.$$

This couple corresponds exactly to the attractor of the trajectories on figure **??**.

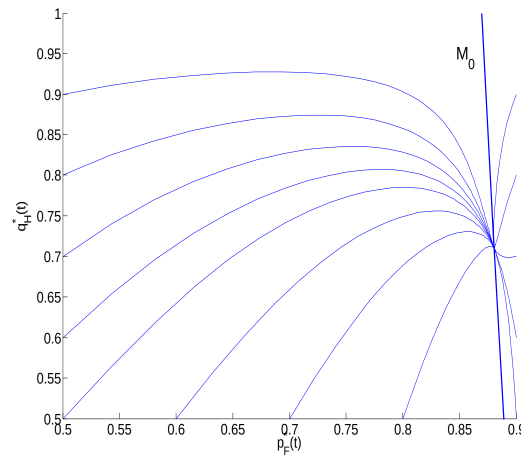

Figure 6.1: Trajectories of the system (S_ϵ) from different starting points and the slow manifold M_0 with $\epsilon = 0.01$.

6.5 Conclusions and perspectives

In this chapter, we have considered a particular type of evolutionary game in which the action of the individual not only determines its immediate fitness but it also impacts his state. The aim is to describe the coupled dynamics of the individual states which is due to the direct control and the dynamics of policies which is determined by the replicator dynamics mechanism. Once we have introduced these combined dynamics, we have proved that any stable rest point corresponds to an equilibrium of the game. We have proposed two methods to obtain the rest points under the assumption that the two dynamics have different time scales. Finally, we discuss how to generalize our framework to any finite number of states and actions, which can lead to several applications of our framework in networking, social networks and complex systems. In perspective, we propose to further investigate the general case to extend our SPcD to the MDEG framework. We are also interested in studying the stability conditions for a rest point of the SPcD to be an ESS profile of our game.